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Preheating of massive fermions after inflation: analytical results

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Abstract: Non-perturbative production of fermions after chaotic inflation has been the object of several studies in the very recent past. However, the results in the most interesting case of production of massive fermions in an expanding Universe were so far known only numerically. We provide very simple and readable analytical formulae, both for the spectra of the created fermions and for their total energy density. Their derivation is closely related to the one adopted for bosons and exploits the fact that the production occurs during very short intervals of non-adiabatical change of the fermionic frequency. Our formulae show the presence of resonance bands if the expansion of the Universe is neglected, and their disappearance when the latter is included. As in the bosonic case, this last effect is due to the stochastic character that the expansion gives to the whole process. Backreaction is considered in the last part of the work. All our analytical results are in excellent agreement with the previous numerical ones in the regime of validity of the latter. However, a more accurate scaling for the energy density of the produced fermions is here found.

Keywords: Cosmology of Theories beyond the SM, Physics of the Early Universe.

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Contents

1. Introduction

One of the key ingredients for our understanding of the early Universe is the mechanism of inflation [1], which constitutes a very elegant solution to several cosmological problems.

Despite of the simplicity of the general idea, the details of the physical processes which govern it are still somehow unclear and matter of intense work. The two main aims of these studies are

(i) to embed inflation in a context more motivated by particle physics (for a review see [2]) and

(ii) to understand the reheating phase which converts the energy density that drives inflation into the matter and radiation that we presently see.

This second issue has been deeply influenced in the last decade by the possibility of particle creation through parametric resonance [3]. The application of this phenomenon to creation of matter after inflation has been called *preheating* in the paper [4], since (with the exception of some very recent versions [5]) it is usually followed by a stage of (ordinary) perturbative reheating.

Preheating of bosons [4, 6] is characterized by a very efficient and explosive creation, due to the coherent effect of the oscillations of the inflaton field. This allows significant production even when single particle decay is kinematically forbidden. It has been very recently noticed [7] that preheating of fermions can also be very efficient despite the production is in this case limited by Pauli blocking.

Parametric creation of spin 1/2 fermions has been the subject of some works in the past. Pure gravitational production has been examined in refs. [8, 9], while creation by an oscillating background field is instead considered in the works [10, 11, 12, 5, 7, 13]. References [10, 11] report results for creation in a Minkowski space. Reference [12] studies the production of massless fermions after a $\lambda\phi^4$ inflation, exploiting the fact that this case can also be reconducted to a static one. In this work, production in a static Universe after chaotic inflation is also considered, and some conjectures on the effects of the expansion are made. Moreover, the full calculation of preheating of massive fermions after chaotic inflation in an expanding Universe has been performed numerically in ref. [7].

These last works had a great impact on the most recent studies. For example, their results turned also useful to the study of gravitinos production at preheating [14]. This issue is particularly important, since gravitinos can easily overclose the Universe (if they are stable) or (if they decay) spoil the successful predictions of primordial nucleosynthesis through photodissociation of the light elements. Gravitinos can be thermally produced during the stage of reheating. To avoid this overproduction, the reheating temperature $T_{\rm RH}$ after inflation cannot be larger than $\sim \left(10^8 - 10^9\right)$ GeV [15]. However, it has been realized [14] that the non-thermal production of helicity $\pm 1/2$ gravitinos (whose equation of motion can be reconducted to the one of an ordinary spin 1/2 Dirac particle) can easily be more efficient than the thermal one, and this in general leads to more stringent upper bounds on $T_{\rm RH}$. Several papers related to the works [14] have recently appeared [16, 17].

Another important implication of preheating of fermions is constituted by leptogenesis, as the work [7] and the related papers [18] show. In this scheme [19], a leptonic asymmetry is first created from the decay of right-handed neutrinos, and then partially converted to baryon asymmetry through sphaleronic interactions. Since leptogenesis is very sensitive to both the mechanism of creation of the heavy neutrinos and to the neutrino mass matrices, it could constitute an interesting link between preheating and the experiments on neutrino oscillations.

Other phenomenological implications of these works appear in refs. [20], with preheating as a possible mechanism for creating superheavy relic particles responsible for the ultrahigh energy cosmic rays, and in ref. [21], where the possible impact of fermions produced *during* inflation on the microwave background anisotropies and on the large-structure surveys is considered. Finally, fermionic production can play an interesting role in hybrid inflationary models [17, 22].

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Due to the large number of these studies, it may be worth to reconsider the basic mechanism of fermionic preheating. In particular, it should be important to give an analytical confirmation to the results of production of massive fermions in the expanding Universe, which are so far known only numerically from the analysis [7]. This is the aim of the present work.

In the next section we revise the basic formalism for preheating of fermions. We consider creation of very massive particles right after chaotic inflation. The coupling of the fermions to the oscillating inflaton gives them a time varying mass. As it is known, this can cause a non-adiabatical change of the frequency of the fermions and their consequent creation. In case of very massive fermions, the non-adiabaticity condition can be satisfied only when their total mass vanishes, and the production occurs at discrete intervals, until the inflaton oscillations become too small for the total fermionic mass to vanish.

In section 3 we derive analytical formulae for the spectra of the fermions after a generic production. Our derivation follows the one developed in ref. [4] for preheating of bosons. It exploits the fact that the production occurs in very short intervals around the zeros of the total fermionic mass: the calculation is made possible from the fact that the occupation number can be considered as constant outside these small regions, and that the expansion of the Universe can be neglected inside them. As a result, the only physical quantities relevant for the creation are the time derivative ϕ' of the inflaton field and the value of the scale factor a at each production. The derivation of the "fermionic counterpart" of the formulae obtained for preheating of bosons in ref. [4] has also been done to a certain extent in the work [21], where the results of a single production during inflation is given. However, when one is interested in the successive productions, a more detailed study is necessary, as our analysis shows. As may be expected, the final results that we obtain closely resemble the ones of the work [4]. What is most surprising is their excellent agreement with the numerical results, as some figures provided manifest.

In section 4 we consider the production in a non-expanding Universe. In this case our analytical formulae considerably simplify and agree with the ones of refs. [11, 12]. In particular, they show the presence of resonance bands which are anyhow limited by Pauli blocking.

In section 5 we study the more interesting case of production in an expanding Universe. As indicated in ref. [12] and as confirmed by the numerical results of the work [7], the creation is now very different with respect to the previous case. The expansion removes the resonance bands and the production (almost) saturates a Fermi sphere up to a maximal momentum. Our analytical results confirm this behavior.

In section 5 we also calculate the total energy density ρ_X of produced fermions, which may be the quantity of most physical relevance. We compare our results with the ones of ref. [7], where it is shown that the final value of ρ_X (normalized to the

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inflaton energy density) scales linearly with the parameter $q \equiv (g^2\phi_0^2)/(4m_\phi^2)$,[1] while it depends very weakly on the fermion bare mass m_X. However, these numerical results are valid only in a limited range for m_X, and it has been wondered if the density ρ_X decreases at values of m_X below this range. Our analytical results give a positive answer in this regard. To see this, a proper average of the analytical formulae must be done, exploiting the fact that the expansion of the Universe gives the production a stochastic character. In this way one can get a "mean" function that interpolates very well between the maxima and the minima of the spectra of produced particles. Again we derived it in close analogy with what is done in the bosonic case, and again the results that we get are in very good agreement with the numerical ones of ref. [7] in the region of validity of the latter.

All this analysis neglects the backreaction of the produced fermions on the evolution of the inflaton field and of the scale factor. Despite the difficulty of a more complete treatment, backreaction effects can be understood at least in the Hartree approximation. This was done numerically in ref. [7]. In section 6 we see that the analytical formulae here provided allow to understand the effects of backreaction observed in the numerical simulations.

2. Production of fermions at preheating

In this section we revise the basic formalism for production of fermions at preheating. Our presentation follows the ones of refs. [7, 21] with the correction of few typos.

We start from the Dirac equation (in conformal time η) for a fermionic field X in the Friedmann-Robertson-Walker background:

$$\left(\frac{i}{a}\gamma^\mu\,\partial_\mu + i\,\frac{3}{2}H\gamma^0 - m\right) X = 0\,, \tag{2.1}$$

where a is the scale factor of the Universe, $H = a'/a^2$ the Hubble rate,[2] and m the total mass of the fermion.

Fermionic production is possible if this last quantity varies non adiabatically with time. This may happen during the coherent oscillations of the inflaton ϕ at reheating in presence of a Yukawa interaction $\phi\bar{X}X$, such that the total mass is given by

$$m(\eta) = m_X + g\phi(\eta)\,. \tag{2.2}$$

We will always consider fermions with very high bare mass m_X. In this case, non adiabaticity can be achieved if the Yukawa interaction is sufficiently strong to make the total mass (2.2) vanish. Fermions are then created whenever the inflaton field crosses the value $\phi_* \equiv -m_X/g$.

[1] In this expression g is the Yukawa coupling between the inflaton and the fermions, ϕ_0 the initial value of the inflaton, and m_ϕ its mass.

[2] Here and in the following prime denotes derivative which respect to η.

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We will study the production after chaotic inflation, that is while the inflaton field coherently oscillates about the minimum of the potential

$$V = \frac{1}{2}\, m_\phi^2 \phi^2\,, \quad m_\phi \simeq 10^{13}\,\mathrm{GeV}\,. \tag{2.3}$$

If one neglects the backreaction of the created particles (this effect will be considered in the last part of the work), then, after few oscillations, the inflaton evolves according to the formula

$$\phi(t) \simeq \frac{M_p}{\sqrt{3\pi}} \frac{\cos\left(m_\phi t\right)}{m_\phi t}\,, \tag{2.4}$$

where t is the physical time. The presence of the t at the denominator of eq. (2.4) shows the damp of the oscillations due to the expansion of the Universe. It thus follows that there exists a final time after which $|\phi| < m_X/g$ and the total mass no longer vanishes, so that the production ends.

To proceed with our analysis, we redefine $\chi = a^{-3/2} X$. Equation (2.1) acquires the more familiar form

$$\left(i\partial\!\!\!/ - am\right)\chi = 0\,. \tag{2.5}$$

We decompose

$$\chi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{-i\mathbf{k}\cdot\mathbf{x}} \sum_r \left[u_r(k,\eta) a_r(k) + v_r(k,\eta) b_r^\dagger(-k)\right]\,, \tag{2.6}$$

with[3]

$$v_r(k) = C\bar{u}_r^T\left(-k\right)\,. \tag{2.7}$$

It is convenient to take the momentum along the third direction $k \equiv k_z$ and to define

$$u_r \equiv \left[\frac{u_+(\eta)}{\sqrt{2}}\psi_r\,, \frac{u_-(\eta)}{\sqrt{2}}\psi_r\right]^T\,, \qquad v_r \equiv \left[\frac{v_+(\eta)}{\sqrt{2}}\psi_r\,, \frac{v_-(\eta)}{\sqrt{2}}\psi_r\right]^T \tag{2.8}$$

with $\psi_+ = \binom{1}{0}$ and $\psi_- = \binom{0}{1}$ eigenvectors of the helicity operator $\sigma\cdot\mathbf{v}/|\mathbf{v}|$. From the normalization adopted in eq. (2.8), one can impose

$$u_r^+ u_s = v_r^+ v_s = \delta_{rs}\,, \qquad u_r^+ v_s = 0\,, \qquad |u_+|^2 + |u_-|^2 = 2\,. \tag{2.9}$$

With this choice, the Dirac equation (2.5) rewrites

$$u_\pm'(\eta) = iku_\mp(\eta) \mp iamu_\pm(\eta)\,, \tag{2.10}$$

[3] We choose

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -i\,\sigma_2 \\ -i\,\sigma_2 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix},$$
$$\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \longrightarrow C = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}.$$

which can be decoupled into[4]

$$u''_{\pm} + \left[\omega_k^2 \pm i(am)'\right] u_{\pm} = 0\,, \qquad \omega^2(\eta) = k^2 + a^2m^2\,. \tag{2.11}$$

Under this decomposition, the hamiltonian of the system

$$H = \frac{1}{a} \int d^3\mathbf{x} \chi^+(x) \left(-i\partial_\eta\right) \chi(x) \tag{2.12}$$

rewrites

$$H = \frac{1}{a} \int d^3k \sum_r \Big\{ E_k\left(\eta\right) \left[a_r^\dagger\left(k\right) a_r\left(k\right) - b_r\left(k\right) b_r^\dagger\left(k\right)\right] + \\ + F_k\left(\eta\right) b_r\left(-k\right) a_r\left(k\right) + F_k^*\left(\eta\right) a_r^\dagger\left(k\right) b_r^\dagger\left(-\,k\right) \Big\}\,, \tag{2.13}$$

where (using the equations of motion)

$$\begin{aligned} E_k &= k\,\mathrm{Re}\left(u_+^* u_-\right) + a\,m\left(1 - u_+^* u_+\right), \\ F_k &= \frac{k}{2}\left(u_+^2 - u_-^2\right) + a\,m\,u_+ u_-\,, \qquad E_k^2 + |F_k|^2 = \omega^2\,. \end{aligned} \tag{2.14}$$

It is always possible to choose an initial configuration with the hamiltonian in the standard (diagonal) form. If we take, for example,

$$u_{\pm}\left(\eta_0\right) = \left(1 \mp \frac{am}{\omega}\right)^{1/2} e^{i\phi}\,, \quad \phi \text{ arbitrary phase}\,, \tag{2.15}$$

we have $E_k(\eta_0) = \omega$, $F_k(\eta_0) = 0$.

However, the evolution equations (2.11) drive F_k different from zero and a diagonal form for H can be recovered only after a Bogolyubov transformation on the creation/annihilation operators

$$\begin{aligned} \hat{a}(\eta, k) &\equiv \alpha_k(\eta) a(k) + \beta_k(\eta) b^+(-k)\,, \\ \hat{b}^+(\eta, k) &\equiv -\beta_k^*(\eta) a(k) + \alpha_k^*(\eta) b^+(-k)\,. \end{aligned} \tag{2.16}$$

While it is immediate to see that the equal time anticommutation relations on both sets $\{a^{(+)}, b^{(+)}\}$ and $\{\hat{a}^{(+)}, \hat{b}^{(+)}\}$ enforce

$$|\alpha_k|^2 + |\beta_k|^2 = 1\,, \tag{2.17}$$

the use of some algebra shows that a diagonalization of the hamiltonian is possible with the choice

$$\alpha = \beta\left(\frac{E + \omega}{F^*}\right), \qquad |\beta|^2 = \frac{|F|^2}{2\,\omega\,(E + \omega)} = \frac{\omega - E}{2\,\omega}\,. \tag{2.18}$$

[4] We do not deal with the analogous equations for $v_{\pm}$, since from condition (2.7) it simply follows $v_+(\eta) = -u_-^*(\eta), v_-(\eta) = u_+^*(\eta)$.

The time varying operators $\hat{a}^{(+)}$ and $\hat{b}^{(+)}$ are employed to define time dependent Fock spaces, each of them built from the zero (quasi)particle state at the time η:

$$\hat{a}(\eta)|0_\eta\rangle = \hat{b}(\eta)|0_\eta\rangle = 0\,. \tag{2.19}$$

Let us consider the state $|0_{\eta_0}\rangle \equiv |0\rangle$ which describes a system with zero particles at the initial time η_0. This is no longer true for generic time $\eta > \eta_0$, since the occupation number at η is defined in terms of the operators which diagonalize the hamiltonian at that moment. The particle density per physical volume $V = a^3$ at time η is indeed given by[5]

$$\begin{aligned} n(\eta) \equiv \langle 0|\frac{N}{V}|0\rangle &= \langle 0|\frac{1}{a^3}\sum_{r=\pm 1}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\hat{a}_r^+(\eta,r)\hat{a}_r(\eta,r)|0\rangle = \\ &= \frac{1}{\pi^2\,a^3}\int dk\,k^2\,|\beta_k|^2\,, \end{aligned} \tag{2.20}$$

which is different from zero whenever the hamiltonian is non diagonal in terms of the initial operators. The occupation number of created fermions is thus given by $n_k = |\beta_k|^2$ and the condition (2.17) ensures that the Pauli limit $n_k < 1$ is respected.

3. Analytical evaluation of the occupation number

In this section we calculate analytically the evolution of the Bogolyubov coefficients (2.16) during the oscillations of the inflaton field after chaotic inflation.

In this derivation, we exploit the fact that, in the regime of very massive fermions we are interested in, the creation occurs only for very short intervals about the points $\phi_* \equiv -m_X/g$ where the total fermionic mass (cfr. eq. (2.2)) vanishes. As the perfect agreement with the numerical results will confirm, this consideration allows one to treat the fermionic production with the same formalism adopted in the bosonic case [4]. While far from the zeros of the total mass m the Bogolyubov coefficients can be treated as constant, whenever ϕ crosses ϕ_* a sudden variation occurs. Since the interval of production is very narrow, one can safely neglect the expansion of the Universe during the production and also linearize the function $\phi(\eta) \simeq \phi_* + \phi'(\eta_*)(\eta - \eta_*)$. As a consequence, the frequency ω defined in eq. (2.11) acquires the form

$$\omega^2 \simeq k^2 + a^2\left(\eta_*\right)\phi'^2\left(\eta_*\right)\;\left(\eta-\eta_*\right)^2\,, \tag{3.1}$$

and the whole calculation strongly resembles the one for scattering through a quadratic potential.

The use of this formalism is very well established in case of production of bosons [4]. For what concerns fermions, it has been recently adopted by Chung et al. [21] for the study of the production during inflation. Fermionic production

[5] Of course the same result is achieved also for antifermions.

during inflation is possible only if the coefficient g of the Yukawa interaction has opposite sign with respect to the inflaton field during inflation, or the total mass (2.2) would never vanish.[6] If this is the case, it is possible to choose the value of g such that the production occurs only once during inflation, while during reheating $|\phi|$ is always too small for having creation. Our derivation is strongly inspired by this work. However, we are interested in couplings for which the production occurs several times during reheating, and it is not at all guaranteed a priori that an analytical approximation may work also in this case. The present analysis not only positively answers to this question, but also provides very simple and explicit formulae valid for arbitrary number of productions.

As may be expected, the final formulae are very similar to the ones found for production of bosons [4]. What is most surprising is the almost perfect agreement between the analytical results and the numerical simulations, as we will show with some figures at the end of this section.

The first step for the derivation of the analytical formulae is to consider asymptotic solutions of eqs. (2.10) and (2.11). We look for solutions valid for ϕ not very close to ϕ_*, where the adiabaticity condition $\omega' \ll \omega^2$ holds.

In this regime, the most general solutions of eqs. (2.10) and (2.11) are

$$\begin{aligned} u_+ &= A\left(1-\frac{am}{\omega}\right)^{1/2} e^{i\int^\eta \omega_k\, d\eta} + B\left(1+\frac{am}{\omega}\right)^{1/2} e^{-i\int^\eta \omega_k\, d\eta}\,, \\ u_- &= -B\left(1-\frac{am}{\omega}\right)^{1/2} e^{-i\int^\eta \omega_k\, d\eta} + A\left(1+\frac{am}{\omega}\right)^{1/2} e^{i\int^\eta \omega_k\, d\eta}\,, \end{aligned} \tag{3.2}$$

with $|A|^2+|B|^2=1$ following from the condition $|u_+|^2+|u_-|^2=2$.

We can always put solutions of eqs. (2.10) and (2.11) into the form (3.2), with A and B in general functions of η. However, in most of the evolution (whenever the adiabaticity condition holds) it is a very good approximation to treat the coefficients A and B as constant.

Substituting the expressions (3.2) into eqs. (2.14), one finds

$$F = 2\omega AB\,, \qquad E = \omega\left(|A|^2-|B|^2\right), \tag{3.3}$$

from which it follows

$$|\beta|^2 = \frac{|F|^2}{2\omega\,(E+\omega)} = |B|^2\,. \tag{3.4}$$

One can thus choose (up to an irrelevant global phase)

$$A \equiv \alpha \longrightarrow B \equiv \beta^*\,, \tag{3.5}$$

where α and β are nothing but the Bogolyubov coefficients we are interested in.

[6] Of course this constraint does not apply to our case, since the inflaton field changes sign after each half oscillation.

Notice that the initial choice $A(\eta_0) = 1,\ B(\eta_0) = 0$ corresponds to the initial condition (2.15) and to the zero particles state chosen in the previous section.

As we have said, these coefficients undergo a sudden change whenever ϕ crosses ϕ_0 and then they stabilize to new (almost) constant values. Our aim is to find the values at the end of the variation in terms of the ones prior to it.

We have not specified the lower limit of the integrals appearing in eqs. (3.2). For present convenience we choose it to be the time η_{*1} of the first production (that is when $\phi = \phi_*$ for the first time).

Let us consider the evolution equation (2.11) near the point η_{*1}. Since for high mass m_X the fermionic production is limited to a very short interval, one can neglect the expansion of the Universe during it and write the equation for $\phi(\eta)$ in a linearized form. We can thus write

$$a\,m(\eta) \simeq a_{*1}\,g\,\phi'_{*1}\,(\eta - \eta_{*\,1})\,, \qquad a_{*1} \equiv a\,(\eta_{*\,1})\,, \qquad \phi_{*1} \equiv \left.\frac{d\,\phi}{d\,\eta}\right|_{\eta_{*1}}. \tag{3.6}$$

Following the notation of [21], we define

$$p \equiv \frac{k}{\sqrt{g|\phi'_{*1}|a_{*1}}}\,, \qquad \tau = \sqrt{g|\phi'_{*1}|a_{*1}}\,(\eta - \eta_{*1})\,. \tag{3.7}$$

In terms of these new quantities, eqs. (2.11) rewrite (dot denotes derivative with respect to τ)[7]

$$\ddot{u}_\pm + \left(p^2 \mp i + \tau^2\right) u_\pm = 0\,. \tag{3.9}$$

The point η_{*1} is thus mapped into the origin of τ and the region of asymptotic adiabaticity is at large $|\tau|$.

In the asymptotic solutions (3.2) we can see the behaviors

$$\begin{aligned}
\left(1 + \frac{am}{\omega}\right)^{1/2} &\longrightarrow \frac{p}{\sqrt{2}\tau}\,,\\
\left(1 - \frac{am}{\omega}\right)^{1/2} &\longrightarrow \sqrt{2}\,,\\
e^{\pm i\int^\eta \omega_k d\eta} &\longrightarrow \left(\frac{2\tau}{p}\right)^{\pm ip^2/2} e^{\pm i\tau^2/2} e^{\pm ip^2/4}\,,
\end{aligned} \tag{3.10}$$

[7] For the production at the moment η_{*n}, when the total mass vanishes for the n-th time, eq. (3.9) must be replaced by

$$\ddot{u}_\pm + \left(p^2 \pm i\,\mathrm{sign}\,(\phi'_{*n}) + \tau^2\right) u_\pm = 0\,. \tag{3.8}$$

The effect of this replacement on the final results is reported below.

for $\tau \to +\infty$, and

$$\begin{aligned} \left(1 + \frac{am}{\omega}\right)^{1/2} &\longrightarrow \sqrt{2}\,, \\ \left(1 - \frac{am}{\omega}\right)^{1/2} &\longrightarrow \frac{p}{-\sqrt{2}\tau}\,, \\ e^{\pm i\int^{\eta}\omega_k d\eta} &\longrightarrow \left(\frac{p}{-2\tau}\right)^{\pm ip^2/2} e^{\mp i\tau^2/2} e^{\mp ip^2/4}\,, \end{aligned} \tag{3.11}$$

for $\tau \to -\infty$.

Equations (3.9) are solved [21] by parabolic cylinder functions $D_\lambda(z)$ [23]. More precisely, the combination that matches the asymptotic solution (3.2) at $\tau \to -\infty$ is[8]

$$\begin{aligned} u_+(\tau) = A^-\sqrt{2}\left(\frac{p}{\sqrt{2}}\right)^{1+ip^2/2} & e^{i\left(\frac{\pi}{4}-\frac{p^2}{4}\right)} e^{-\pi p^2/8} D_{-1-ip^2/2}\left(-\left(1+i\right)\tau\right) + \\ + B^-\sqrt{2}\left(\frac{p}{\sqrt{2}}\right)^{-ip^2/2} & e^{ip^2/4} e^{-\pi p^2/8} D_{ip^2/2}(-(1-i)\tau)\,. \end{aligned} \tag{3.12}$$

In the above expression, A^- and B^- denote the values of the Bogolyubov coefficients before ϕ crosses ϕ_*, while the functions which multiply them are *exact* solutions of the *linearized* equation (3.9). The analytical approximation consists in considering them as solutions of the true evolution equation (2.11). For $\tau \to +\infty$ it is convenient to rewrite the solution (3.12) in terms of two different parabolic cylinder functions[9] [23]

$$\begin{aligned} u_+(\tau) = & \left[A^-\frac{\sqrt{\pi}pe^{-\pi p^2/4}}{\Gamma\left(1+ip^2/2\right)} e^{i\left(\frac{\pi}{4}-\frac{p^2}{2}+\frac{p^2}{2}\ln\frac{p^2}{2}\right)} + B^- e^{-\pi p^2/2}\right] \times \\ & \times \left\{\sqrt{2}\left(\frac{p}{\sqrt{2}}\right)^{-ip^2/2} e^{ip^2/4} e^{-\pi p^2/8} D_{ip^2/2}((1-i)\tau)\right\} + \\ & + \left[-A^-\,e^{-\pi p^2/2} + B^-\,\frac{\sqrt{\pi}\,p\,e^{-\pi p^2/4}}{\Gamma\left(1-i\,p^2/2\right)}\, e^{-i\left(\frac{\pi}{4}-\frac{p^2}{2}+\frac{p^2}{2}\ln\frac{p^2}{2}\right)}\right] \times \\ & \times \left\{\sqrt{2}\left(\frac{p}{\sqrt{2}}\right)^{1+i\,p^2/2} e^{i\left(\frac{\pi}{4}-\frac{p^2}{4}\right)} e^{-\pi p^2/8} D_{-1-ip^2/2}((1+i)\tau)\right\}. \end{aligned} \tag{3.13}$$

In this new expression, the functions within curly brackets correspond to the asymptotic forms at $\tau = +\infty$ of the two terms of the solution (3.2). The coefficients in front of them give thus the new Bogolyubov coefficients in terms of the old ones.

All this derivation can be easily generalized when productions at successive zeros of the total mass m are considered. The only important points are

[8] We deal only with the function u_+, since the study of u_- leads to the same results.

[9] This rewriting is always possible since eq. (3.9) has only two linearly independent solutions.

(i) a difference in the values of the scale factor a and of the derivative ϕ' at different η_{*i}'s,

(ii) a change of sign in the transfer matrix (the one which gives the new coefficients in terms of the old ones) whenever ϕ crosses ϕ_* from below to above (cfr. the footnote just before eq. (3.9)), and

(iii) the phase $e^{\pm i\int \omega_k d\eta}$ which accumulates between η_{*1} and the η_{*i} considered.

Putting all this together, one has

$$\begin{aligned}
\begin{pmatrix} \alpha_n \\ \beta_n^* \end{pmatrix} &= \begin{pmatrix} F_n & H_n \\ -H_n^* & F_n^* \end{pmatrix} \begin{pmatrix} \alpha_{n-1} \\ \beta_{n-1}^* \end{pmatrix} \qquad \text{for } n \text{ odd}\,, \\
H_n &\longleftrightarrow -H_n \qquad\qquad\qquad\qquad \text{for } n \text{ even}\,, \qquad (3.14)
\end{aligned}$$

where a_n, β_n are the values of the Bogolyubov coefficients after the n-th production, and where

$$\begin{aligned}
F_n &= \sqrt{1-e^{-\pi p_n^2}}\, e^{i\left(-\frac{\pi}{4} - \arg\Gamma(ip_n^2/2) + \frac{p_n^2}{2}\ln(p_n^2/2) - p_n^2/2\right)}\,, \\
H_n &= e^{-\pi p_n^2/2} e^{-2i\int_{\eta_{*1}}^{\eta_{*n}} \omega_k\, d\eta}\,, \qquad |F_n|^2 + |H_n|^2 = 1\,. \qquad (3.15)
\end{aligned}$$

We remind that $p_n = k/\sqrt{g|\phi'_{*n}|a_{*n}}$.

If one starts with no fermions at the beginning, one may choose $\alpha_0 = 1, \beta_0 = 0$. Then, applying successive times the "transfer" matrix (3.14), one can get the spectrum of fermions produced after every η_{*n}.

Of course our calculation reproduces the result

$$N_1 = |\beta_1|^2 = e^{-\pi p_1^2} \qquad (3.16)$$

reported in ref. [21].

We numerically integrated the evolution equations for $u_\pm$, ϕ, and the scale factor a.[10] The results obtained with the analytical expression (3.14) are always in very good agreement with the numerical ones. Just to give a couple of examples, we present here two cases at different regimes (we show them only with illustrative purpose, and the values of the parameters chosen have no particular importance). In figure 1 we present the spectrum of the fermions after two productions, that is after one complete oscillation of the inflaton field. In analogy with the bosonic case, we measure the strength of the coupling inflaton-fermions with the quantity $q \equiv g^2\phi_0^2/(4m_\phi^2)$, where $\phi_0 \simeq 0.28M_p$ is the value of the inflaton at the beginning of reheating. In figure 1 we choose $q = 10^8$, while we fix the bare fermion mass to be $m_X = 100m_\phi$. In figure 2 we show instead the resulting spectrum after 7 productions in the case $q = 10^4$, $m_X = 4m_\phi$.

[10] Our starting point is at $\phi(0) = 0.28M_p$, short after inflation, $\phi'(0) = -0.15M_p m_\phi$ (as follows from a numerical evaluation of the inflaton alone during inflation), and $a(0) = 1$.

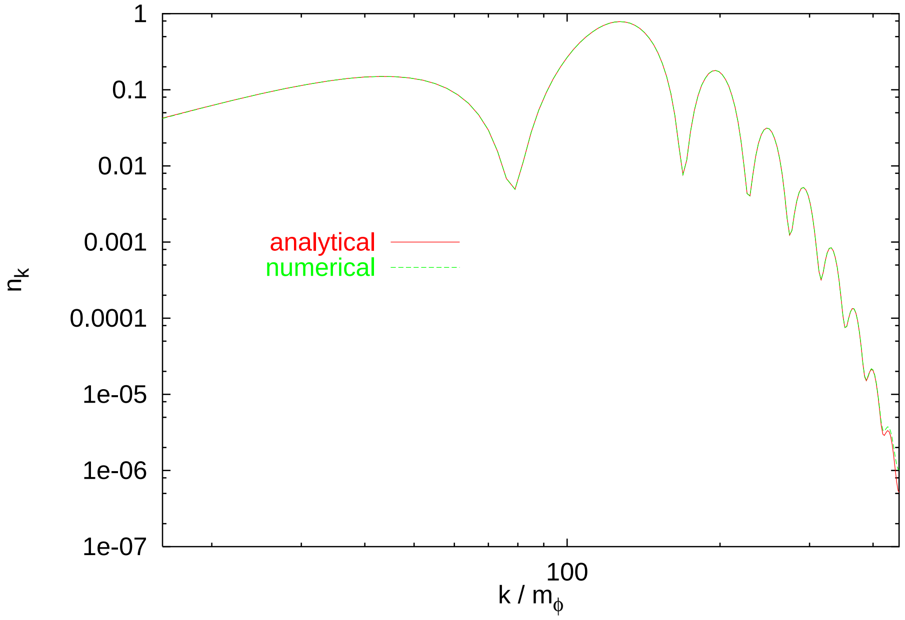

Figure 1: Spectrum of the fermions after two productions for $q = 10^8$ and $m_X = 100m_\phi$. The expansion of the Universe is taken into account.

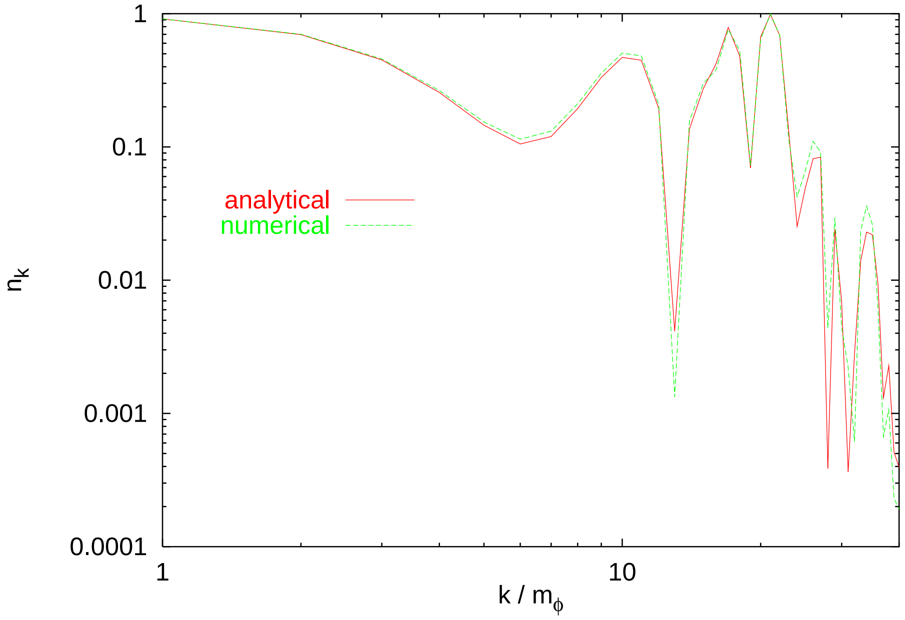

Figure 2: Spectrum of the fermions after seven productions for $q = 10^4$ and $m_X = 4m_\phi$. The expansion of the Universe is taken into account.

4. Production in a non-expanding Universe

In the bosonic case, the study of the non-perturbative production in a non-expanding Universe has proven very useful in understanding the effects of the production. It is shown in ref. [4] that the bosonic wave function satisfies the Mathieu equations, whose solutions are characterized by resonance bands (in momentum space) of very "explosive" and efficient production. It is then shown that, due to the expansion of the Universe, modes of a given comoving momentum k cross several resonance bands during the evolution. This gives the creation the stochastic character described in the work [4]. In ref. [12] it is understood that an analogous behavior occurs also for fermions. The expansion is expected also in this case to spoil the clear picture of distinct resonance bands. This fact may help the transfer of energy to fermions, since the resonance bands in the fermionic case are anyhow limited by the Pauli principle. The expansion allows thus new modes to be occupied, and the production is no longer limited to the regions of resonance. In ref. [12] it is stated that the production should then almost completely fill the whole Fermi sphere up to a maximal momentum $k_{\rm max}$. This behavior is confirmed by the numerical results of the work [7]. In this section we will see that our analytical formulae can reproduce the resonance bands, while in the next one we will discuss the effects of the expansion of the Universe.

Let us consider the matrices (we drop the index n in the matrix elements since all the p_n's have now the same value p)

$$M = \begin{pmatrix} F & G \\ -G & F^* \end{pmatrix}, \qquad T_1 = \begin{pmatrix} e^{i\,\vartheta_1^2} & 0 \\ 0 & e^{-i\vartheta_1^2} \end{pmatrix}, \qquad T_2 = \begin{pmatrix} e^{i\vartheta_2^3} & 0 \\ 0 & e^{-i\vartheta_2^3} \end{pmatrix}, \tag{4.1}$$

with $G \equiv e^{-\pi p^2/2}$ and $\vartheta_i^j \equiv \int_{\eta_{*i}}^{\eta_{*j}} \omega_k d\eta$ (F was defined in the previous section).

Without the expansion of the Universe, the inflaton field has the periodic evolution

$$\phi(\eta \equiv t) = \phi_0 \cos\left(m_\phi \eta\right), \tag{4.2}$$

and all the ϑ_i^j (3.14) are hence sums of ϑ_1^2 and ϑ_2^3 (remember the η_{*i} are the moments at which the total fermionic mass vanishes).

After the generic n-th complete oscillation one thus has

$$\begin{pmatrix} \alpha_{2n} \\ \beta_{2n}^* \end{pmatrix} = \begin{pmatrix} e^{-\,i\,\vartheta_1^{2n+1}} & 0 \\ 0 & e^{i\vartheta_1^{2n+1}} \end{pmatrix} T_2 M^T T_1 M \cdots T_2 M^T T_1 M \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.3}$$

with the combination $\hat{O} \equiv T_2 M^T T_1 M$ repeated n times.

One has thus simply to study the eigenvalue problem for $\hat{O}$ (notice $\det \hat{O} = 1$). This operator has the form

$$\begin{aligned} \hat{O} &= \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}, \\ A &= F^2 e^{i\left(\vartheta_1^2+\vartheta_2^3\right)} + G^2 e^{-i\left(\vartheta_1^2-\vartheta_2^3\right)}, \\ B &= FG e^{i\left(\vartheta_1^2+\vartheta_2^3\right)} - F^* G e^{-i\left(\vartheta_1^2-\vartheta_2^3\right)} \end{aligned} \tag{4.4}$$

and its eigenvalues are $\lambda_{1,2} = e^{\pm i\Lambda}$ with $\cos\Lambda = \mathrm{Re}\, A$.

Rewriting the initial condition $\binom{1}{0}$ in terms of the eigenvectors of $\hat{O}$ and substituting in formula (4.3), one gets the number of produced fermions

$$N_n = |\beta_n|^2 = \frac{|B|^2}{1-(\mathrm{Re}\ A)^2}\sin^2(n\Lambda) = \frac{|B|^2}{\sin^2\Lambda}\sin^2(n\Lambda) \tag{4.5}$$

after the complete n-th oscillation.[11]

We notice the presence of the envelope function

$$\tilde{E} \equiv \frac{|B|^2}{1-(\mathrm{Re}\,A)^2} = \frac{1-|A|^2}{1-(\mathrm{Re}\,A)^2} \tag{4.6}$$

which modulates the oscillating function $\sin^2(n\Lambda)$.

With increasing n this last function oscillates very rapidly and can be at all effects averaged to 1/2. One is thus left with the envelope function which shows the presence of resonance bands. The resonance bands occur where A is real and $\tilde{E} \to 1$. It is easy to understand that their width exponentially decreases with increasing momenta k. To see this, let us consider the behavior of $\tilde{E}$ at high momenta. In this regime, the function A is given by

$$A \simeq (1-e^{-\pi p^2})e^{i\phi_A}\,, \qquad p = \frac{k}{\sqrt{g|\phi'_*|}}\,, \tag{4.7}$$

where the phase ϕ_A can be read from eq. (4.4). Near to the points where $\cos\phi_A = 1$ the envelope function behaves like[12]

$$\tilde{E} \simeq \frac{1-(1-e^{-\pi p^2})^2}{1-(1-e^{-\pi p^2})^2\cos^2\phi_A} \simeq \frac{1}{e^{\pi p^2}(1-\cos\phi_A)+1}\,. \tag{4.8}$$

The width of the band can be defined as the distance between the two successive points at which $\tilde{E} = 1/2$. From the last expression it follows that the difference between the phases of A in these two points is given by $\Delta\alpha = 2\sqrt{2}e^{-\pi p^2}$. Since the most rapidly varying term which contributes to the phase of A is $(p^2/2)\log(p^2/2)$, the width of the band can be thus estimated to be

$$\Delta p \simeq \frac{2\sqrt{2}}{p\log(p^2/2)}\,e^{-\pi p^2}\,. \tag{4.9}$$

We show in figure 3 the envelope of the produced fermions in a static Universe for the parameters $q = 10^6$ and $m_X = 100\,m_\phi$. The peaks occur where A is real and it is confirmed that their width decreases very rapidly at increasing momenta. Due to the fact that the last peaks plotted are indeed very sharp, the resolution of figure 3 does not allow to see their top at $n_k = 1$.

[11] In ref. [12] it is said that it is possible to extract the average over one period of the occupation number from the knowledge of the solution of eq. (2.11) with initial conditions $u_+(\eta_0) = 1, u'_+(\eta_0) = 0$. The matching of this procedure with our formulae (3.2) and (4.4) gives the average time evolution (in our notation) $\bar{N}(t) = |B|^2\,\sin^2[(\pi-\Lambda)t/T]/\sin^2[\pi-\Lambda]$, where T is the period of one oscillation. Clearly, this result coincides with our eq. (4.5) at any given complete oscillation.

[12] A completely analogous behavior occurs where $\cos\phi_A = -1$.

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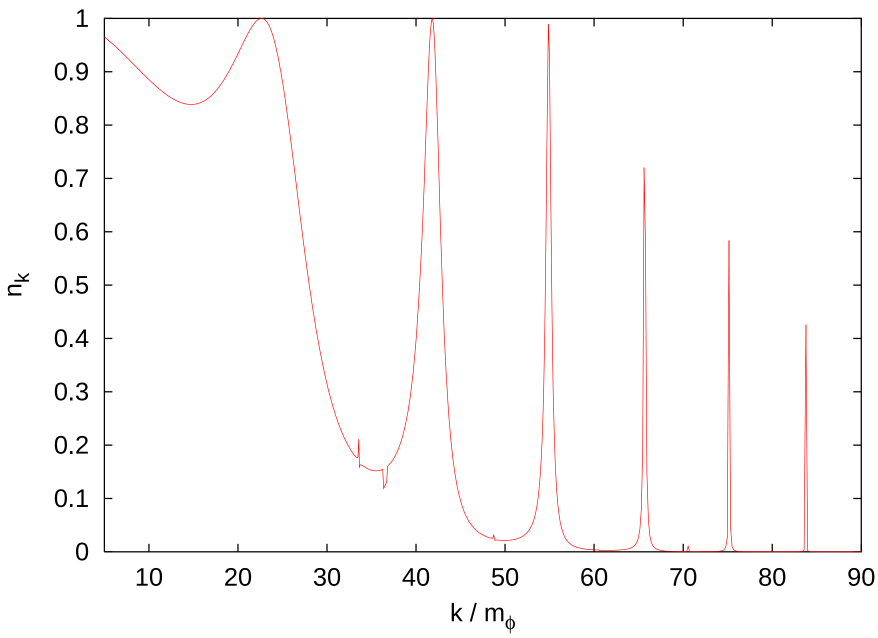

Figure 3: Envelope of the spectrum of the produced fermions in a static Universe. The physical parameters are $q = 10^6$ and $m_X = 100\, m_\phi$.

5. Expansion taken into account

As stated in the previous section, the resonance bands disappear when the expansion of the Universe is taken into account. In this section we will show how the occupation number varies when a non-vanishing Hubble parameter is considered. As we have seen in section 2, eqs. (3.14) and (3.15) give a very good agreement with the numerical results. On the other hand, the presence of phases in eq. (3.15) makes the exact analytical treatment of the occupation number impossible. Now, the same observation made in the bosonic case [4] turns very useful also to us: the phases in eq. (3.15), when the expansion of the Universe is taken into account, are not correlated among themselves. As a result, the final spectra present several high frequency oscillations about some average function. The positions of the peaks of these oscillations depend on the details of the phases. However, the "mean" function can be understood in a surprisingly easy way. Our problem can be treated as one customary does when dealing with the "random walk". Imagine one has to calculate the quantity

$$S \equiv |A_1 + A_2 + A_3 + \cdots + A_n|^2\,, \tag{5.1}$$

where the A_i are complex numbers with random phases. The "random walk recipe" indicates that the best estimation of the above quantity is achieved by summing the squares of the terms A_i, since the mixed products average to zero. The chaoticity of the final spectra for n_k suggests that this may also be true in our case, and this is confirmed by comparison with the numerical results.

With this method, eqs. (3.14) and (3.15) turn into the much simpler relations

$$\begin{pmatrix} |\alpha_n|^2 \\ |\beta_n^*|^2 \end{pmatrix} = \begin{pmatrix} |F_n|^2 & |H_n|^2 \\ |H_n|^2 & |F_n|^2 \end{pmatrix} \begin{pmatrix} |\alpha_{n-1}|^2 \\ |\beta_{n-1}^*|^2 \end{pmatrix}, \tag{5.2}$$

where we remember

$$|F_n|^2 = 1 - e^{-\pi p_n^2}, \qquad |H_n|^2 = e^{-\pi p_n^2}, \qquad |F_n|^2 + |H_n|^2 = 1\,. \tag{5.3}$$

Assuming no fermions in the initial state, and applying n times this formula, it is easy to see that the occupation number after the n-th production is given by

$$N_n(k) = \frac{1}{2} - \frac{1}{2}\prod_{i=1}^{n}\left(1 - 2e^{-\pi p_i^2}\right). \tag{5.4}$$

A similar result holds for preheating of bosons, cfr. [4] where the idea of averaging on almost random phases is first introduced. In the bosonic case, one can exploit the fact that, due to the high efficiency of the production, the occupation number after the $(n+1)$-th creation is (almost) proportional to the occupation number after the n-th one:

$$N_{n+1}\left(k\right) \simeq \left(1 + 2e^{\pi\kappa_n^2}\right) N_n\left(k\right), \tag{5.5}$$

where the quantity κ_n is analogous to our parameter p_n. This simplification is not possible in our case, since the Pauli principle forbids N_n to be sufficiently high. However our final result, eq. (5.4), is also cast in a very simple and immediate form.

The validity of eq. (5.4) is confirmed by our numerical investigations, as we show here in one particular case. In figure 4 we compare the behavior of the "mean" function for the spectra with respect to the numerical one. We choose the physical parameters to be $q = 10^4, m_X = 4m_\phi$, and we look at the results after the 7-th production (this corresponds to the choices made in figure 2). We see that the "mean" function interpolates very well between the maxima and the minima of the numerical spectrum, and that it can indeed be considered as a very good approximation of the actual result.

This is confirmed by figure 5, where we plot (for the same values adopted in figure 4) the quantity $n_k k^2$ rather than the occupation number alone. This quantity is of more physical relevance when one is interested in the total energy transferred to the fermions, since the total number density of produced fermions is (apart from the dilution due to the expansion of the Universe, that will be considered only in the final result)

$$N_X = \frac{2}{\pi^2}\int dk\, k^2\, n_k\,. \tag{5.6}$$

As shown in the plot, the result for N_X in the numerical and in the approximated case are in very good agreement.

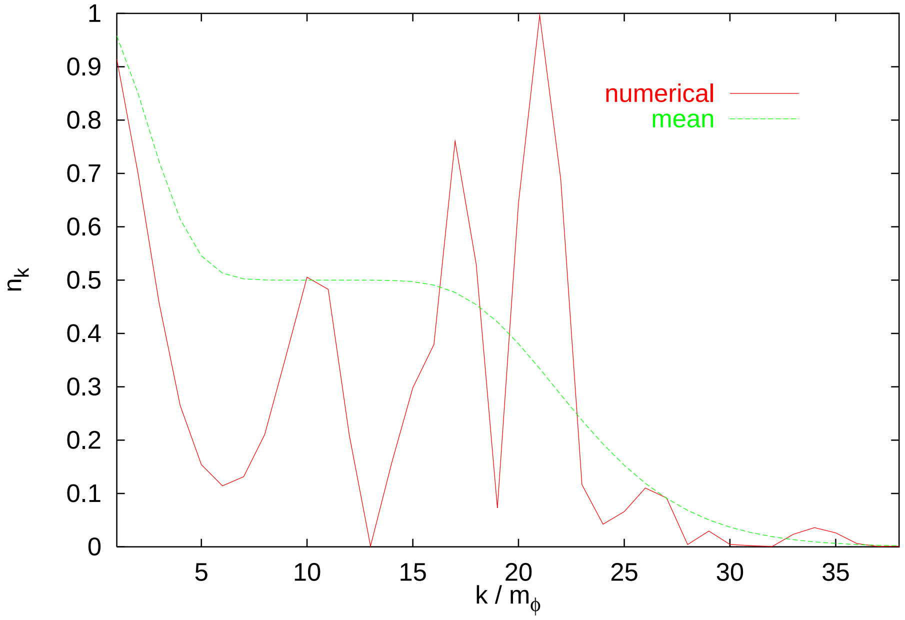

Figure 4: Comparison between the numerical spectrum and the "mean" function after seven productions for $q = 10^4$ and $m_X = 4m_\phi$.

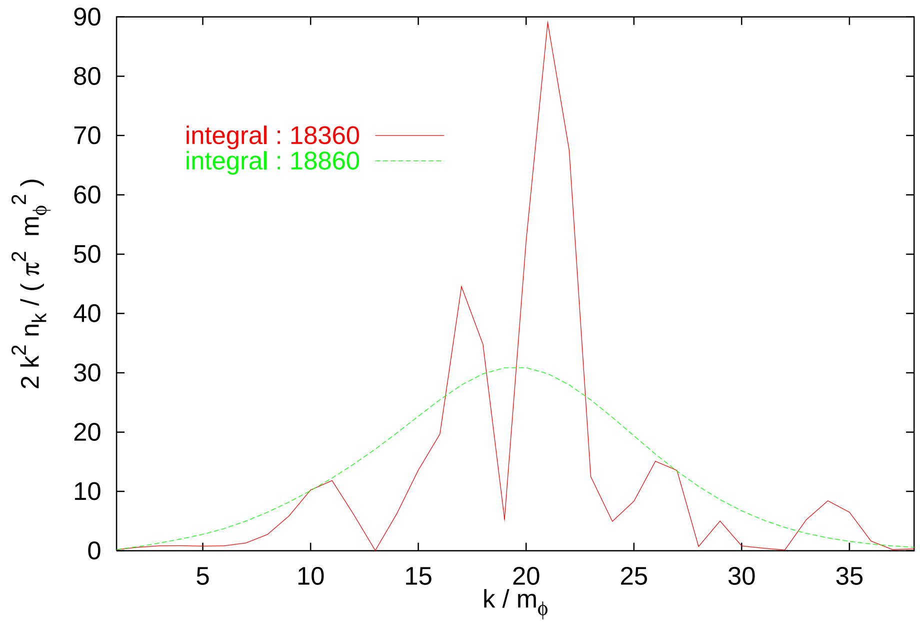

Figure 5: As in figure 4, but with the quantity $k^2 n_k$ plotted.

After checking the validity of the approximation given by the "mean" function, we adopt it to understand how the production scales when different values of the parameters q and m_X are considered.

Equation (5.4) allows us to give an analytical estimate of the total amount of energy stored in the fermions after the n-th production, and in particular after that the whole process of non-perturbative production has been completed. Notice that all the dependence on the physical parameters is in the coefficients

$$z_i \equiv \frac{k}{(\pi^{1/2} p_i)}\,, \tag{5.7}$$

where k is the comoving momentum, and the only part we have to determine are the numbers

$$z_i^2 = \frac{2\sqrt{q}}{\pi} a\left(\eta_{*i}\right) \frac{\left|\phi'\left(\eta_{*i}\right)\right|}{\phi_0}\,. \tag{5.8}$$

Here, and in what follows, we express the dimensionful quantities in units of m_ϕ, the inflaton mass, apart from the inflaton field ϕ that is given in units of M_p.

We also introduce the ratio $R \equiv 2q^{1/2}/m_X$. This quantity is the most relevant, since it determines the zeros of the total fermionic mass and thus the values of the z_i's. Indeed, from eq. (2.2) we see that the total mass vanishes for $\phi_* = -\phi_0/R$. It is convenient to study the production in terms of the two independent parameters q and R (rather than q and m_X) since, at fixed R, all the spectra are the same provided we rescale $k \propto q^{1/4}$ (cfr. eq. (5.8)).

One can now proceed in two different ways, and we devote the next two subsections to each of them. First, one can evolve the equations for the inflaton field alone and find numerically the values z_i for given q and R. Inserting these values in eq. (5.4) one can get final values for the production which, as we have reported, are very close to the numerical ones. This method allows to get results which average the actual ones, and it has the advantage of being much more rapid than a full numerical evolution.

Alternatively, one can proceed with a full analytical study in order to understand the results given by the first semi-analytical method.

5.1 Semi-analytical results

In this subsection we evaluate the analytical formula (5.4), taking the coefficients z_i from a numerical evolution of the inflaton field. As we have said, this method gives results which are very close to the numerical ones, due to the fact that the "mean" function interpolates very well the actual spectra of the produced fermions.

The first thing worth noticing is that, for each choice of q and R, the maximum z_i occurs at about half of the whole process of non-perturbative creation. It thus follows that fermions of maximal comoving momenta will be mainly produced at the half of the process. Our semi-analytical evaluations support this idea: the total energy stored in the created fermions increases slowly during the second part of the preheating. To see this, we show in figure 6 the numerical results for the quantity $N_X = 2\int dk\, k^2 n_k/\pi^2$ as function of the number of production. We fix the physical

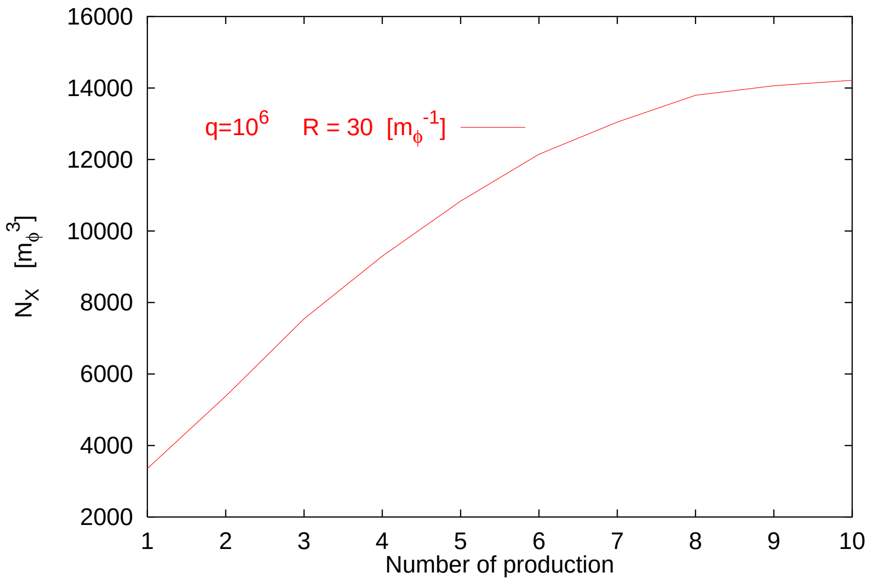

Figure 6: Growth of N_X with the number of productions for $q = 10^6$ and $R = 30$.

parameters to be $q = 10^6$ and $R = 30$, that is $m_X = 67\, m_\phi$. With this choice, the total mass vanishes 10 times, and figure 5 shows the results after each step of this production. We see indeed that the final productions are less efficient than the previous ones.

From eq. (5.4) it is also possible to show the evolution of the spectra with the number of productions. We do this in figure 7. We choose the same parameters as in figure 6, and we show the results after each complete oscillation of the inflaton field (that is after each two productions). We observe that the production rapidly approaches a step function in the momentum space, i.e. there exists a maximum momentum below which Pauli blocking is saturated (notice that the value 1/2 follows from the average understood in the "mean" function), and above which $n_k \simeq 0$. The fact that the last productions do not contribute much to the total energy is also confirmed.[13]

We now turn our attention to the total energy transferred to fermions after the whole preheating process is completed. We fix the parameter q to the value 10^6 and we investigate how the total integral N_X changes with different values for the parameter R.[14] The results are shown in figure 8, for R ranging from 5 to 10000. For the last value the total fermionic mass changes sign more than 3700 times and a full numerical evaluation would appear very problematic. This can be done in our case, thanks to the analytical expression (5.4) found, and our results extend the validity region of the previous full numerical study [7].

[13] The behavior at small k is inessential, since this region does not significantly contribute to the total energy.

[14] As reported, the scaling of the final result with q at fixed R is simply understood from eq. (5.8).

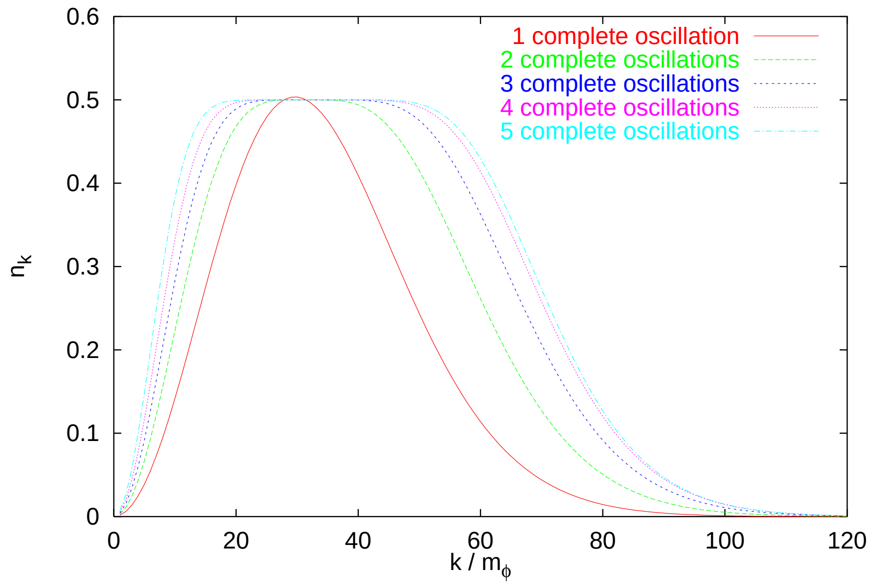

Figure 7: Evolution of the spectrum of produced fermions with the number of productions for $q = 10^6$ and $R = 30$.

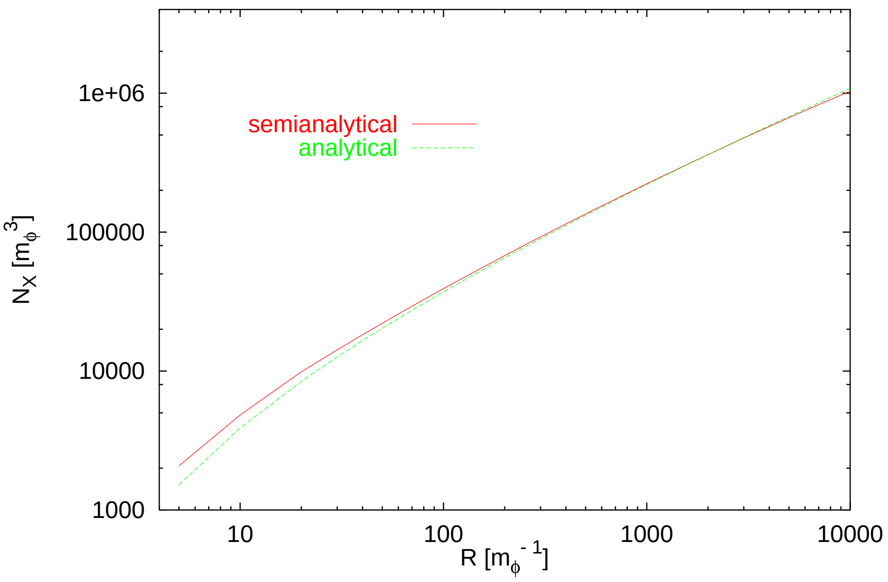

Figure 8: Comparison between the analytical and the semi-analytical results for N_X, at fixed $q = 10^6$.

In figure 8, the results of our semi-analytical method are also compared to the full analytical ones of the next subsection. This comparison will be discussed below.

From the scaling of N_X with R just reported, it is now easy to estimate the total energy transferred to fermions for generic values of q and R. We are interested in comparing our results to the numerical ones of the work [7]. To do so, we consider the

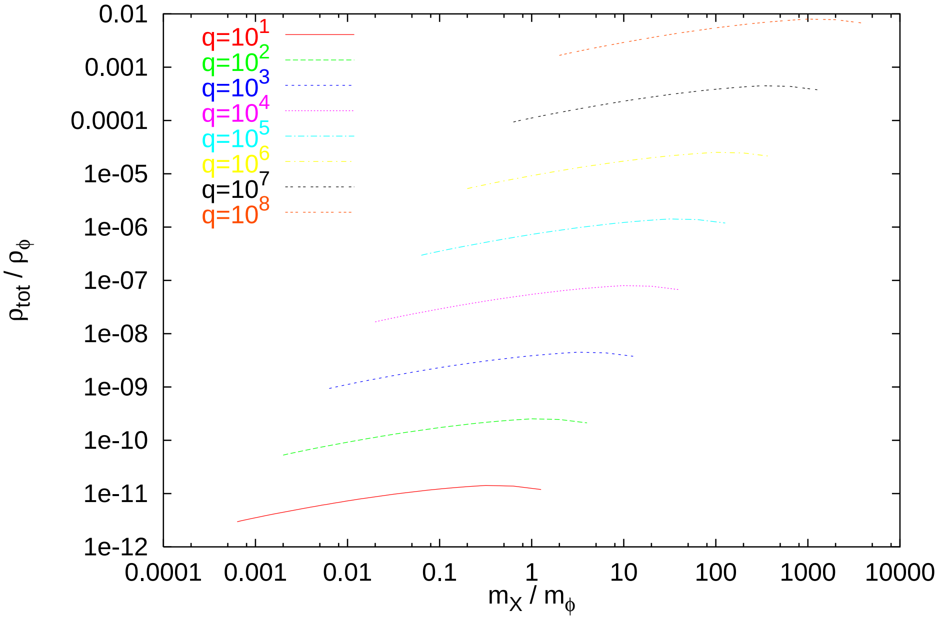

Figure 9: Total final energy density produced (normalized to the inflaton one) for different values of q and m_X.

ratio between the energy density given to fermions and the one in the inflaton field[15]

$$\frac{\rho_X}{\rho_\phi} = \frac{2m_X}{\pi^2} \int dk\, k^2 n_k \cdot \frac{2}{m_\phi^2\, \phi_0^2} \,. \tag{5.9}$$

We present our results in figure 9. We report them in terms of q and m_X, in order to have an immediate comparison to the analogous figure 1 of ref. [7].

In figure 9, for any fixed q, the greatest plotted value for m_X corresponds to the choice $R = 5$. We are not interested in extending this limit since we know that for greater m_X (actually for values greater than the bound $m_X \sim \sqrt{q}/2$) the production suddenly stops. The smallest value plotted for m_X (at any fixed value q) corresponds instead to $R = 10000$, that is to considering more than 3700 productions in the numerical evaluation of eq. (5.4).

Our final values are in good agreement with the ones of figure 1 of ref. [7] in the regime of validity of the latter. The numerical results reported in that figure exhibit small fluctuations about an average function $\rho_X(m_X)$. Our results give this average function. This was expected, since the expression that we integrated, eq. (5.4), interpolates between the maxima and the minima of the numerical spectra.

[15]This ratio should be calculated at a time $t_{\rm end}$ at the end of preheating, when the total fermionic mass stops vanishing. Thus in the denominator of eq. (5.9) the comoving momentum k should be replaced by the physical one $p = k/a$, with a scale factor of the Universe at $t_{\rm end}$. Analogously, the value ϕ_0 of the beginning of reheating should be replaced by the one at $t_{\rm end}$. However, both the replacements cancel out in the ratio, since both ρ_X and ρ_ϕ redshift as energy densities of matter.

The numerical results of ref. [7] have a smaller range of validity than the ones here presented. This occurs because the numerical evolution of that work is limited to the first 20 oscillations of the inflaton field, and so fermionic production has not come to its end for small values of m_X. We confirm that at high values of m_X (actually at small values of R for any given q) the production depends very weakly on m_X. In addition, our results show a decrease of the energy transferred to fermions for smaller values of m_X. This behavior will be explained in details in the next subsection.

5.2 Analytical results

We want now to show that all the results presented in the previous section can be also achieved with a full analytical study of eq. (5.4).

First of all, we have to estimate the quantities z_i given in eq. (5.8). To do this, it is more convenient to work in terms of the physical time t: after the first few oscillations, the inflaton evolution is very well approximated by the expression (remember t is expressed in units of m_ϕ^{-1}, while ϕ in units of M_p)

$$\phi(t) \simeq \frac{1}{\sqrt{3\pi}} \frac{\cos(t)}{t} \,. \tag{5.10}$$

The scale factor of the Universe follows the "matter-domination" law, and it is well approximated by $a(t) = t^{2/3}$.

The values t_{*i} are determined by the condition of vanishing of the total mass of the fermions, that is, by making use of eq. (5.10),

$$\cos(t_{*i}) = -\frac{t_{*i}}{RA} \,, \tag{5.11}$$

where we remind $R \equiv (2\sqrt{q})/m_X$. The parameter $A \equiv (\sqrt{3\pi}\,\phi_0)^{-1}$ is of order one and will not play any special role in what follows. Notice that the last production occurs at $t \simeq RA$.

Hence, keeping only the dominant contribution to the derivative of ϕ with respect to the physical time, we get the expression

$$z_i^2 \cong \frac{2\sqrt{q}}{\pi} R^{1/3} A^{4/3} \left(\frac{t_{*i}}{RA}\right)^{1/3} \sqrt{1 - \left(\frac{t_{*i}}{RA}\right)^2} \,, \tag{5.12}$$

where we can assume $t_{*i} \simeq i\pi$.

Equation (5.12) exhibits a very good agreement with the numerical evaluation of the same quantity. It also shows that the maximal value for z_i is reached at $t_{*i} = AR/2$, that is, at half of the whole process of non-perturbative creation. This was anticipated in the previous section, where we showed that the most of the fermionic production occurs in the first part of preheating.

Starting from eq. (5.12) we can also calculate the number density of produced particles

$$N_X(q,R) = \frac{2}{\pi^2}\int dk\, k^2 N(k)\,, \tag{5.13}$$

where $N(k)$ is obtained from eq. (5.4) with $n = n_{\max} = (RA)/\pi$.

For R big enough, the product in eq. (5.4) can be written as the exponential of an integral. Thus, we obtain

$$\begin{aligned} N_X = {} & \frac{m_X}{4\pi^3}\left(\frac{2A^{4/3}}{\pi}\right)^{3/2} q^{3/4}R^{1/2}\times \\ & \times \int_0^\infty dt\, t^2 \left\{1-\exp\left[\frac{R\,A}{\pi}\int_0^1 dy \log\left|1-2\exp\left(-\frac{t^2}{y^{1/3}\sqrt{1-y^2}}\right)\right|\right]\right\}, \end{aligned} \tag{5.14}$$

with the substitution $t = k\cdot\sqrt{\pi/(2A^{4/3}q^{1/2}R^{1/3})}$.

The integral in dy which appears in eq. (5.14) cannot be calculated analytically. Anyhow, we can approximate it by

$$\int_0^1 dy \log\left|1-2\exp\left(-\frac{t^2}{y^{1/3}\sqrt{1-y^2}}\right)\right| \simeq g\,(t)\log\left|1-2e^{-1.5t^2}\right|, \tag{5.15}$$

where $g(t)$ is a function of order one that, for our purposes, can be approximated by a constant c in the range $0.5 \lesssim c \lesssim 1$.

Hence, the integrand within curly brackets in eq. (5.14) rewrites

$$1-\left|1-2e^{-1.5t^2}\right|^{c\frac{RA}{\pi}}. \tag{5.16}$$

In the large-R limit this function approximates a step function, which evaluates to one for

$$\sqrt{\frac{\pi\log 2}{2RAc}} \lesssim t \lesssim \sqrt{\log\left(\frac{2RAc}{\pi\log 2}\right)} \tag{5.17}$$

and to zero for the remaining values of t.

Since the quantity (5.16) is proportional to the occupation number n_k, our analytical calculation confirms the usual assumption that, after few oscillations, the fermionic production saturates the Fermi sphere up to a given maximum momentum $k_{\max}$. This is also shown in the previous figure 7. However, our derivation gives a different scaling for $k_{\max}$ with respect to the previous literature [12, 7]. Indeed, from eq. (5.17) it follows (apart from proportionality factors)

$$k_{\max} \propto \frac{q^{1/3}}{m_X^{1/6}}\sqrt{\log\left(\frac{q^{1/2}}{m_X}\right)}, \tag{5.18}$$

which is however quite close to the result given in [7]. We will comment more about the origin of the scaling (5.18) in the conclusions.

From eqs. (5.14) and (5.17), we obtain the final expression for the number density of the fermions created during the whole process

$$N_X(q,R) = \frac{1}{3\pi^2}\left(\frac{2A^{4/3}}{1.5\pi}\right)^{3/2} q^{3/4} R^{1/2} \left[\log\left(\frac{4Ac}{\pi\log 2}\frac{q^{1/2}}{m_X}\right)\right]^{3/2} . \tag{5.19}$$

We can now go back to figure 8, where this last equation (called "analytical" in the figure) is compared to the result with the semi-analytical method of the previous subsection. In plotting eq. (5.19) we chose $c = 0.78$ for the numerical factor involved. As we can see, the final results achieved with the two methods are in very good agreement with each other, thus confirming the validity of the formula (5.19).[16]

Rewriting eq. (5.19) in terms of q and m_X we can draw some conclusions. First, apart from a logarithmic correction, the scaling of the total energy

$$\rho_X \propto m_X N_X \propto q m_X^{1/2} \left[\log\left(\frac{4Ac}{\pi\log 2}\frac{q^{1/2}}{m_X}\right)\right]^{3/2} \tag{5.20}$$

is linear in q, as generally expected [7]. The dependence of ρ_X on m_X requires some more care: the threshold value for m_X is given by the condition $R \sim 4$, that is,[17]

$$(m_X)_{\rm th} \sim \frac{\sqrt{q}}{2} . \tag{5.21}$$

For values of m_X not much smaller than $(m_X)_{\rm th}$, the total energy depends very weakly on m_X, this result being in agreement with the numerical evaluation given in [7]. On the other hand, for values of m_X much smaller than $(m_X)_{\rm th}$, the factor $m_X^{1/2}$ starts to dominate, and we expect it to determine the scaling of the total energy when $m_X \to 0$.

6. Backreaction

The results presented so far have been achieved neglecting the backreaction of the produced fermions on the evolution of the inflaton field and on the scale factor. This is a common approximation, since a more complete treatment (especially an analytical one) of the whole phenomenon is a very difficult task. However, the effects of the backreaction can be understood to a good degree of accuracy in the Hartree approximation.

For what concerns preheating of fermions, the Hartree approximation consists in taking into account the term

$$g\langle \bar{X} X\rangle \tag{6.1}$$

[16] The small discrepancy between the two curves can be attributed to the fact that c is not exactly constant.

[17] The number 4 comes from the fact that the value of the inflaton at its first minimum is $\phi \simeq -0.07 M_p$, while at beginning $\phi_0 = 0.28 M_p$.

into the evolution equations for the inflaton and the scale factor. The equation for the field ϕ thus rewrites (in physical time)

$$\ddot{\phi} + 3H\dot{\phi} + m_\phi^2 \phi + g\langle \bar{X} X \rangle = 0 \,. \tag{6.2}$$

The study of this effect has been performed numerically in ref. [7], where it is shown that backreaction starts to be important for $q > q_b \sim 10^8 - 10^{10}$. Figure 7 of that work shows how the evolution of the inflaton field ϕ is modified when backreaction is considered and q is sufficiently high. First, one observes that the amplitude of the oscillations of the inflaton is very damped already after the first production. This effect is the most obvious one, since the term (6.1) takes into account the decay of the inflaton into fermion-antifermion pairs, while in its absence the equation for ϕ considers only the damp due to the expansion of the Universe. The second feature that emerges from the evolution performed in ref. [7] is that at the beginning the field ϕ does not oscillate about the minimum of the potential $V = m_\phi^2 \phi^2/2$, but about the point ϕ_* where the total fermionic mass vanishes. Moreover, the frequency of these oscillations is higher than m_ϕ.

These last two effects are due to the change in the effective potential for ϕ induced by the term (6.1), and disappear when the quantity $\langle \bar{X} X \rangle$ is decreased by the expansion of the Universe. Their net effect is to render the whole mechanism of preheating more efficient, since the rise in the frequency of the oscillations of ϕ increases the number of productions of fermions. In ref. [7] it is indeed shown that for $q = 10^8$ the total production is about 5% larger than the one without backreaction, while for $q = 10^{10}$ the increase is about 50%.

We now briefly study the evolution of the inflaton ϕ under eq. (6.2) by means of the analytical results presented above. We show that even a very approximate analysis confirms the numerical result that indicates in $q \simeq 10^8 - 10^{10}$ the threshold above which backreaction should be considered.

To begin on, the term (6.1) needs to be normal ordered. Doing so, one gets [7]

$$\langle \bar{X} X \rangle = \frac{2}{(2\pi a)^3} \int d^3\mathbf{k} \left(1 + \frac{ma}{\omega} - |u_-|^2 \right) . \tag{6.3}$$

In terms of the Bogolyubov coefficients, this quantity evaluates to

$$\langle \bar{X} X \rangle = \frac{4}{(2\pi a)^3} \int d^3\mathbf{k} \left[|\beta|^2 \frac{am}{\omega} + \mathrm{Re} \left(\alpha\beta \frac{k}{\omega} \, e^{2i\int \omega \, d\eta} \right) \right] . \tag{6.4}$$

We see that $\langle \bar{X} X \rangle$ vanishes for $\beta = 0$. This is obvious, since backreaction starts only after fermions are produced. Some approximations can render eq. (6.4) more manageable. First, we notice that the oscillating term in the exponential averages to very small values the integral of the second term in square brackets. Second, we see that $k \ll |am|$ where β_k is significantly different from zero. From both these considerations, the integrand in eq. (6.4) can be approximated (up to the sign of m) by the occupation number $|\beta|^2$, so that the whole effect is (approximatively) proportional to the number of produced fermions.

The numerical results of ref. [7] show that, for the values of q for which the backreaction is to be considered, its effects can be seen already in the first oscillation of the inflaton field. Since we are only interested in estimating the order of magnitude of q_b, we thus concentrate on the first oscillation of ϕ, neglecting the expansion of the Universe in this short interval.

With all these approximations, eq. (6.2) rewrites

$$y'' + y + 10^{-12} q^{5/4} \frac{m}{|m|} = 0 \,, \tag{6.5}$$

where we have rescaled $y \equiv \phi/\phi_0$ and we remind that the time is given in units of m_ϕ^{-1}.

This equation is very similar to the one obtained in the bosonic case [4]. The last term changes sign each time $m = 0$ and, when sufficiently high, forces the inflaton field to oscillate about the point ϕ_* at which the total fermionic mass vanishes. It is also responsible for the increase of the frequency of the oscillations. To see this, we assume that this term dominates over the second one of eq. (6.5) and we solve it right after the first fermionic production at the time t_{*1}. In the absence of the second term, eq. (6.5) is obviously solved by a segment of parabola until the time $t_{*2} \simeq t_{*1} + 2|y'(t_{*1})|/(q^{5/4}10^{-12})$ at which m vanishes again and the last term of eq. (6.5) changes sign. As long as the second term of eq. (6.5) can be neglected, the inflaton evolution proceeds along segments of parabola among successive zeros of the mass m. The "time" duration of these segments is expected to be of the same order of the first one, since the successive fermionic productions balance the decrease of $\langle \bar{X} X \rangle$ due to the expansion of the Universe.[18]

The period of these oscillations can thus be roughly estimated to be $T \sim 2(t_{*2} - t_{*1})$. We see that, for $q \gtrsim 10^9$, this period is smaller than the one that the inflaton oscillations would have neglecting backreaction. Since this increase of the frequency is the main responsible for the higher fermionic production, the result $q_b \sim 10^9$ can be considered our estimate for the value of q above which backreaction should be taken into account.

This result, although obtained with several approximations, is in agreement with the numerical one of ref. [7].

7. Conclusions

Preheating of fermions and bosons present several analogies. In both cases, especially when one is interested in the production of very massive particles, the creation occurs

[18] However, after the first part of the process, the production looses its efficiency and the expansion of the Universe dominates. As we have said, the term $\langle \bar{X} X \rangle$ can then be neglected and the inflaton starts oscillating about the minimum of the tree level potential.

for very short intervals of time, during which the frequency of the particles varies non adiabatically and their occupation number cannot be defined. This strong similarity can be exploited to extend to the fermionic case the formalism developed in ref. [4] for the analytical study of preheating of bosons. This was first done in ref. [21], where the result for the particles created in the first production is reported. When one wants to consider the successive productions, a more detailed analysis is necessary. This has been the object of the first part of the present work. The formulae that we obtained, valid for an arbitrary number of productions, give a very good agreement with the numerical results, as some examples provided manifest.

Another strong similarity between the two phenomena is that in both cases the production would occur only through resonance bands (in momentum space) was not for the expansion of the Universe. For what concerns fermions, this feature was first studied in ref. [11, 12], where production in a static Universe is considered. As it is well known, the expansion of the Universe removes the resonance bands and, as a result, this allows the fermionic production to saturate the whole Fermi sphere up to a maximal momentum $k_{\rm max}$. Our analytical results confirm both the presence of the resonance bands in the static case (where our formulae considerably simplify) and their disappearance when the expansion is considered.

Rather than the detailed form of the spectrum, a quantity which may be of more physical interest is the total energy of the produced particles. The analogy with the bosonic case turns useful also in this regard. In the final spectra one notices the presence of some maxima and minima, whose exact position is determined by the detailed knowledge of the phases of the Bogolyubov coefficients (adopted in the analytical derivation) after each production. However, due to the expansion of the Universe, these phases are effectively uncorrelated among themselves. This gives the production a stochastic character which is responsible for the disappearance of the resonance bands. Averaging over these phases, it is possible to get an analytical function for the mean occupation number. For what concerns preheating of bosons, the results are simplified by the fact that, in the region of very efficient production, the mean occupation number after a given production is proportional to the mean occupation number of the previous one. This assumption is not possible in the fermionic case, since the Pauli principle prevents the occupation number to exceed 1. However, it is remarkable that also for fermions the final result can be cast in a very simple and readable form.

The analytical formula for the mean occupation number confirms the saturation of the Fermi sphere (times the factor $1/2$ which comes from the average) up to a momentum $k_{\rm max}$. In order to get this quantity, it is sufficient to know the derivative of the inflaton field and the value of the scale factor at the points where the productions occur. These values can be calculated numerically evolving the equations for the inflaton field alone (at least when backreaction is neglected), or analytically. The former possibility is much more rapid than a full numerical computation, since one does not have to consider the equations for the fermions. Moreover, its results agree very well with the full numerical ones [7]. Thanks to the increase in the rapidity, this evaluation (that is called "semi-analytical" in the present work) allows to get results with a more extended range of validity with respect to the previous full numerical ones.

The results provided in this way can also be achieved with a full analytical study. From both these methods, we deduce that the ratio ρ_X/ρ_ϕ scales as

$$\frac{\rho_X}{\rho_\phi} \sim q m_X^{1/2} \left[\log \frac{q^{1/2}}{m_X}\right]^{3/2} , \tag{7.1}$$

up to $(m_X)_{\rm th} \sim \sqrt{q}/2$.

For m_X not too smaller than $(m_X)_{\rm th}$ this scaling is in good agreement with the one $\rho_X/\rho_\phi \sim q$ of the numerical work [7]. However, we see that the density of produced fermions decreases at smaller values of m_X.

We would like to conclude our work showing that the scaling (7.1) can be also achieved from very immediate considerations. As we said, preheating of fermions saturates a Fermi sphere in momentum space up to a maximal momentum $k_{\rm max}$. From the analytical formula (5.2), we notice that at high momenta k the occupation number is well approximated by

$$N_n(k) \simeq \sum_{i=1}^{n} e^{-k^2/z_i^2} , \tag{7.2}$$

where we remember $z_i \propto q^{1/4} a^{1/2}(\eta_{*i}) |\phi'(\eta_{*i})|^{1/2}$.

In this last equation, we replace all the parameters z_i with a mean value $\bar{z}$, so that $N_n \sim n \exp(-k^2/\bar{z}^2)$. The scaling of $\bar{z}$ with the physical parameters q and m_X follows from the scaling of all the z_i. The maximal momentum $k_{\rm max}$ is thus expected to scale as the quantity $z_i (\log n)^{1/2}$. Considering now the evolution of the inflaton field in physical time t, we notice that both the number n of productions and the times t_{*i} at which they occur are proportional to the parameter $R = q^{1/2}/(2m_X)$. Moreover, we see that the z_i's scale as

$$z_i \propto q^{1/4} a_{*i} \left[\frac{d\phi}{dt}|_{*i}\right]^{1/2} \propto q^{1/4} t_{*i}^{2/3} \frac{1}{t_{*i}^{1/2}} \propto q^{1/4} R^{1/6} . \tag{7.3}$$

We thus get $k_{\rm max} \propto q^{1/4} R^{1/6} [\log R]^{1/2}$, from which (remember $\rho_X \simeq m_X k_{\rm max}^3$) the scaling (7.1) simply follows.

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